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Interaction analysis among players in games

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Outline

1. *n*-person cooperative games
2. Interaction
3. *n*-person cooperative game with fuzzy coalitions
4. Interaction among players in games with fuzzy coalitions.
5. Multicriteria decision making and game theory

1. *n*-person cooperative games

Game Theory contains:

1.Non-Cooperative game (competence)

2.Cooperative game(cooperation)

Published paper:

- **Non-cooperative game:**

  Xiaohui Yu, Qiang Zhang.A study on Nash equilibrium of fuzzy noncooperative games. Fuzzy sets and Systems[J]. Under the third review.

- **Cooperative game:**

a. Shujin Li, Qiang Zhang. The measure of interaction among players in games with fuzzy coalitions. Fuzzy sets and Systems[J]. 159(2008) 119-137.

b. Shujin Li, Qiang Zhang. A simplified expression of the Shapley value for fuzzy game. European Journal of Operational Research[J]. To appear

c. Shujin Li, Qiang Zhang. A method of allocation among the partners in revised coalition structures and Topsis method for ordering the structures. Operational Transactions[J]. 2007,2:99-106.

d. Tong Li, Qiang Zhang. Risk analysis in the supplier investment:An application of fuzzy measure and integrals.European Journal of Operational Research[J]. Under review.

- Cooperative game, also called Coalitional game, is a competent decision analysis model for players to achieve the maximum profit through cooperation with each other.
- In cooperative game, players consider how to form the coalition and how to allocate the wealth of the coalition.
- Express the *n*-person cooperative game in mathematical form:

Def 1: For n-person cooperative game $G=(N, v)$, we shall let $N=\{1, 2, \dots, n\}$ be the set of all players, v is a real-valued characteristic function defined on the subsets of N. An nonempty subset of N is called a coalition, satisfies

(1) $v(\varnothing)=0$;

(2) For any $S, T \subset N$, if $S \cap T = \varnothing$, then

$$v(S \cup T) \geq v(S) + v(T)$$

The characteristic function v in (2) satisfies superadditivity.

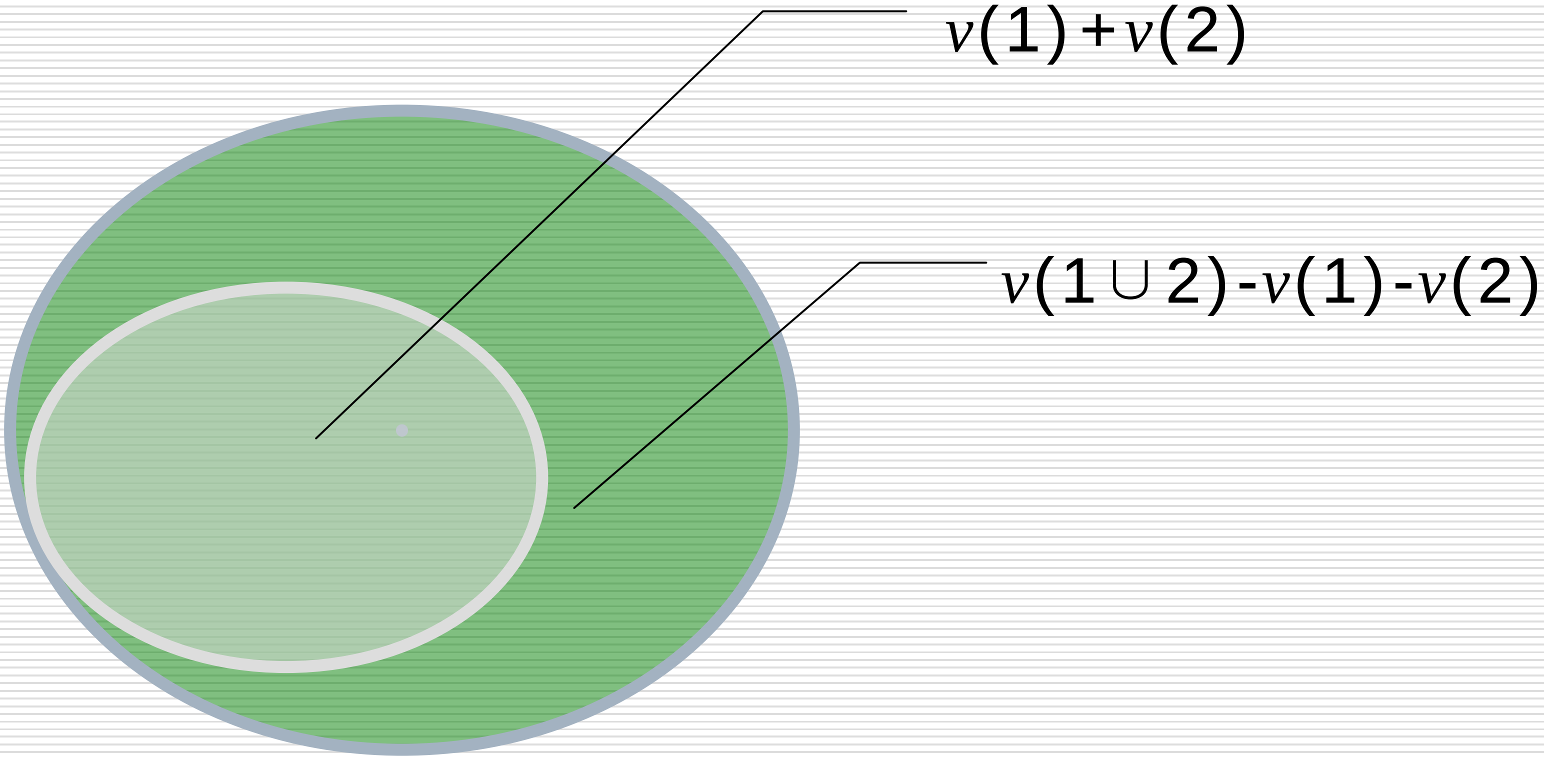
- In traditional cooperative games, the superadditivity of v is a basic assumption.
- $v(S)$ can be interpreted as the cost savings that the members of S obtain when they cooperate. When v stands for the cost function, it becomes:

For any $S, T \subset N$, if $S \cap T = \varnothing$, then

$$v(S \cup T) \leqslant v(S) + v(T)$$

The characteristic function v satisfies subadditivity.

Power of cooperation



- In *n*-person cooperative game, the allocate scheme of the income of the coalition is called solution.

The essential problem to the solution is how to fairly divide the extra earnings of the coalition.

- There have been several solution concepts:
  - Core
  - Stable sets
  - Kernel
  - Bargaining sets
  - Shapley value
  - Banzhaf value

Core and Shapley value are two important solutions.

Shapley value

$$\varphi_i(v) = \sum_{S \subset N \setminus \{i\}} \frac{s!(n-s-1)!}{n!}[v(S \cup \{i\}) - v(S)]$$

Why do players *i* and *j* have different allocation amount?

How do players cooperate to make the extra earnings?

2.Interaction

- ◆ Interaction among players in games
  - M. Grabisch, M. Roubens, An axiomatic approach to the concept of interaction among players in cooperative games.
  - I. Kojadinovic, An axiomatic approach to the measurement of the amount of interaction among criteria or players
- ◆ Interaction among criteria in decision making
  - M. Grabisch, The application of fuzzy integrals in multicriteria decision making[J], European J. Oper. Res.1996, (89): 445-456.

•Interaction among players

- Marginal contribution of player *i*: (*i-derivative*)

$$\delta_i v(T) = v(T \cup \{i\}) - v(T)$$

- Marginal contribution of two players *i* and *j* in the presence of coalition T

$$\delta_{ij} v(T) = v(T \cup \{i, j\}) - v(T \cup \{i\}) - v(T \cup \{j\}) + v(T)$$

- Simultaneous interaction among players in *S*

$$\delta_S w(T) = \sum_{L \subseteq S} (-1)^{s-l} w(T \cup L), \forall S \subseteq N, T \subseteq N \setminus S$$

- Shapley interaction index between two players *i* and *j* in the game $w \in G^N$:

$$I(w,ij) = \sum_{T \subseteq N \setminus ij} \frac{(|N| - |T| - 2)!|T|!}{(|N| - 1)!} \delta_{ij} w(T)$$

- Shapley index of a coalition $S \subseteq N$ in the game $w \in G^N$

$$I_{sh}(w,S) = \sum_{T \subseteq N \setminus S} \frac{(|N| - |T| - |S|)!|T|!}{(|N| - |S| + 1)!} \delta_S w(T)$$

3. ***n*-person cooperative game with fuzzy coalition**

Fuzzy cooperative game have three embranchments:

1. n-person cooperative game with fuzzy coalitions
2. *n*-person cooperative game with fuzzy worth
3. *n*-person cooperative game with fuzzy coalitions and fuzzy worth

Def 2: A fuzzy coalition variable of player set N is a n-dimension variable $x_i = (x_1, x_2, ..., x_n) \in [0,1]^N$

where the ith coordinate x_i is referred to as the variable of participation level of player i.

e.g. LP=(1/0.6, 2/0.5, 3/0.7) implies that player 1, 2 and 3 join in the fuzzy coalition LP with participation level 0.6, 0.5, 0.7 respectively.

4. Interaction among players in games with fuzzy coalitions.

- Shujin Li, Qiang Zhang. The measure of interaction among players in games with fuzzy coalitions. Fuzzy sets and Systems[J]. 159(2008) 119-137.

Abstract:

a. Define the derivative among participation levels of players. to describe simultaneous interaction.
b. Establish the interaction index among participation levels.
c. Propose the integral index of (absolute) interaction among levels.
d. Construct the Shapley total integral index of interaction among players.

- Shujin Li, Qiang Zhang. A simplified expression of the Shapley value for fuzzy game. European Journal of Operational Research[J]. To appear.

Achievement:

a. Discuss the rationality of the existed two Shapley value for cooperative game with fuzzy coalition through interaction among two participation levels.

Preliminary

- $x^{\#}$: the set of fuzzy coalition variable created by x

$$x^{\#} = \{t \mid t_i = x_i \quad or \quad t_i = 0 \quad for \quad each \quad i \in N\}$$

- $\delta_{x_i} v(t)$: the set of x_i -derivative of v in the presence of $t(t \in (x - x_i e^i)^{\#})$

$$\delta_{x_i} v(t) = v(t \vee x_i e^i) - v(t)$$

Achievements:

- $\delta_{x_i x_j} v(t)$:the x_i, x_j -derivative of in the presence of $t(t \in (x - x_i e^i - x_j e^j)^{\#})$, is defined by

$$\delta_{x_i x_j} v(t) = v(t \vee x_i e^i \vee x_j e^j) - v(t \vee x_i e^i) - v(t \vee x_j e^j) + v(t)$$

- $\delta_{LP} v(t)$:the LP -derivative of in the presence of $t(t \in (x - x_i e^i - x_j e^j)^{\#})$, is defined by

$$\delta_{LP} v(t) = \delta_{x_i} (\delta_{LP \setminus x_i} v(t))$$

- Remark:

a. $\delta_{LP}v(t)\big|_{x=s}>0$, there exists positive simultaneous interaction among levels of LP in the presence of t based on s

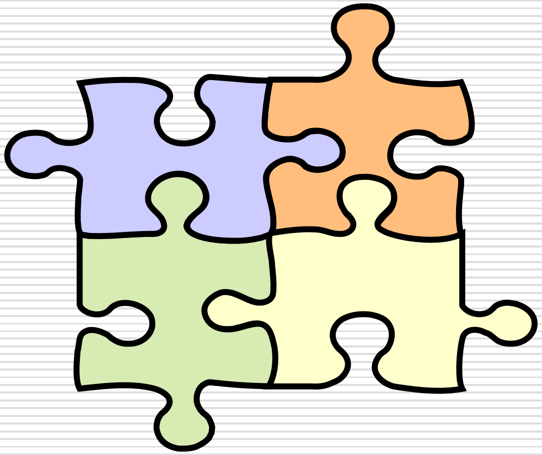
b. $\delta_{LP}v(t)\big|_{x=s}<0$, there exists negative simultaneous interaction among levels of LP in the presence of t based on s

c. $\delta_{LP}v(t)\big|_{x=s}=0$, there is no simultaneous interaction among levels of LP in the presence of t based on s

- Shapley interaction index among the levels:

$$I_{sh}(v,LP)=\sum_{LT\subseteq Lx\backslash LP}\frac{(|N|-|LT|-|LP|)!\,|LT|!}{(|N|-|LP|+1)!}\delta_{LP}v(t)$$

- $I_{sh}(v,LP)$ can be seen as an average measure of the simultaneous interaction among levels of LP



- In order to characterize the average absolute interaction among levels of LP , we define the Shapley absolute interaction index among levels of LP

$$AI_{sh}(v,LP)=\sum_{LT\subseteq Lx\backslash LP}\frac{(|N|-|LT|-|LP|)!|LT|!}{(|N|-|LP|+1)!}|\delta_{LP}v(t)|$$

- It is apparent, when $AI_{sh}(v,LP)\big|_{x=s}=0$ then in the presence of any $t\in(x-\sum_{x_j\in LP}x_j e^j)^{\#}$ there is no interaction among levels of LP based on s

•Mutual independence among levels based on *s*

The levels of LP are said to be marginally mutually independent in the presence of *t* based on fuzzy coalition *s*, if for any $LK \subseteq LP$, there is

$$\delta_{LK} v(t)\big|_{x=s} = 0$$

where $t \in (x - \sum_{x_i \in LP} x_i e^i)^{\#}$

Remark: The marginally mutually independent has indicated that the simultaneous interaction $\delta_{LP} v(t)\big|_{x=s}$ =0 could not prove the marginal mutually independent of the players in *LP*, it should based on the subsets of LP

- Integral index of interaction among levels of *LP* (where the characteristic function is continuous)

The interaction among levels of *LP* with respect to players $M \subseteq N \setminus id(LP)$ is defined by

$$B(v, LP, M) = \int_{[0,1]^m} \delta_{LP} v(t) dLM$$

The Shapley integral index of interaction among levels of *LP* is defined by

$$II_{sh}(v, LP) = \sum_{M \subseteq N \setminus id(LP)} \frac{(n-m-|LP|)!m!}{(n-|LP|+1)!} B(v, LP, M)$$

5.Multicriteria decision making and game theory

- Interaction between criteria

  Tools: Fuzzy measure and integrals

- Choquet integral (Aggregation operator)
- Shapley value (Real importance of the criteria)
- Shapley Interaction index (Interaction analysis among criteria)

- The Shapley value could represent the true importance of the criteria.

$$\varphi_i(v) = \sum_{S \subset N \setminus \{i\}} \frac{s!(n-s-1)!}{n!}[v(S \cup \{i\}) - v(S)]$$

Remark: In game theory, Shapley value can be interpreted as a kind of average value of the contribution of player *i* alone in all coaltions. In decision making, the worth of a coalition of criteria is the importance of the coalition, its importance to make alone the decision(without the remaining criteria)

- The Shapley interaction index $I_{sh}(v,ij)$ could be applied for the analysis of the redundant, substitute, independent correlation among criteria.

a. When $I_{sh}(v,ij)>0$, criteria i and j are redundant
b. When $I_{sh}(v,ij)<0$, criteria i and j are substitute
c. When $I_{sh}(v,ij)=0$, criteria i and j are independent

Example: The director of a high school intends to find out the best student balanced on science and literature. The school is more "scientifically" than literary oriented, more importance is attributed to mathematics and physics.

Student	Mathematics	Physics	Literature	Global evaluation (weighted mean)
A	18	16	10	15.25
B	10	12	18	12.75
C	14	15	15	14.62

The director is not satisfied with the result, according to him, C is better than A. After the interaction analysis, we could find out:

	Shapley value	Shapley Interaction index		
		Mathematics	Physics	Literature
Mathematics	0.35	0	-0.02	0.04
Physics	0.35	-0.02	0	0.03
Literature	0.3	0.04	0.03	0

When apply the Choquet integral as the aggregator:

Student	Mathematics	Physics	Literature	Global evaluation (Choquet integral)
A	18	16	10	13.9
B	10	12	18	13.6
C	14	15	15	14.9

- Tong Li, Qiang Zhang. Risk analysis in the supplier investment: An application of fuzzy measure and integrals. European Journal of Operational Research[J]. Under review.

Thank you!